Enlargements of filtrations and path decompositions at non stopping times

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A CLASS OF REMARKABLE SUBMARTINGALES (II): ENLARGEMENTS OF FILTRATIONS

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ABSTRACT. Azéma associated with an honest time L the supermartingale $Z_t^L = \mathbb{P}[L > t | \mathcal{F}_t]$ and established some of its important properties. This supermartingale plays a central role in the general theory of stochastic processes and in particular in the theory of progressive enlargements of filtrations. In this paper, we shall give an additive characterization for these supermartingales, which in turn will naturally provide many examples of enlargements of filtrations. In particular, we use this characterization to establish some path decomposition results, closely related to or reminiscent of Williams' path decomposition results.

1. INTRODUCTION

The most studied family of random times, after stopping times, are ends of optional sets, or honest times, which we shall always denote by L in this paper. A very powerful technique for studying such random times is the progressive enlargement of filtrations. The theory of progressive enlargements of filtrations was introduced independently by Barlow ([8]) and Yor ([32]), and further developed by Jeulin and Yor ([16, 14, 13, 33]). It has many applications in various parts of Probability Theory: path decompositions for some diffusion processes ([13], [20]), mathematical models of default times and insider trading in mathematical finance ([12]), the Skorokhod embedding problem ([15]), probabilistic inequalities ([16],[22]), or new proofs of well known results, such as Pitman's theorem (see [33], chapter XII).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, satisfying the usual assumptions, and L the end of an $(\mathcal{F}_t)$ optional set Γ, i.e:

$$L = \sup\{t : (t, \omega) \in \Gamma\}.$$

The main idea is to consider the larger filtration

$$\mathcal{F}_t^L = \mathcal{G}_{t+}, \quad \text{with} \qquad \mathcal{G}_t \equiv \mathcal{F}_t \vee \sigma\{L \wedge t,\},$$

which is the smallest right continuous filtration which contains $(\mathcal{F}_t)$ and which makes L a stopping time, and then to see how martingales of the smaller filtration are changed when considered as stochastic processes of the

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larger one. One process plays an essential role in this theory, namely the supermartingale:

$$Z_t^L = \mathbb{P}\left(L > t | \mathcal{F}_t\right), \tag{1.1}$$

associated with L by Azéma in [1], and chosen to be càdlàg. An $(\mathcal{F}_t)$ local martingale (M_t), is a semimartingale in the larger filtration $\left(\mathcal{F}_t^L\right)$ and decomposes as:

$$M_t = \widetilde{M}_t + \int_0^{t\wedge L} \frac{d\langle M, Z^L\rangle_s}{Z_{s_-}^L} - \int_L^t \frac{d\langle M, Z^L\rangle_s}{1 - Z_{s_-}^L}, \tag{1.2}$$

where $\left(\widetilde{M}_t\right)_{t\geq 0}$ denotes a $\left(\left(\mathcal{F}_t^L\right), \mathbb{P}\right)$ local martingale. One limitation of this formula is that it is not easy to compute the supermartingale Z^L for a given L; in fact not so many examples are known (see [13, 33]). Hence, it would be useful to dispose of a characterization result which would help us produce examples of honest times and their associated supermartingales.

For simplicity, and because in practical applications they are always satisfied, we make the following assumptions throughout this paper, which we call the (**CA**) conditions:

(1) all $(\mathcal{F}_t)$-martingales are **c**ontinuous (e.g: the Brownian filtration).
(2) the random time L **a**voids every $(\mathcal{F}_t)$-stopping time T, i.e. $\mathbb{P}[L = T] = 0$.

Remark 1.1. Under the conditions (**CA**), the optional and the predictable sigma fields are equal and the supermartingale (Z_t^L) is continuous.

One of the aims of this paper is to characterize the supermartingales (Z_t^L). In [20], we gave a multiplicative characterization for the supermartingales (Z_t^L), while here we shall adopt an additive approach (Doob-Meyer decomposition). The paper is organized as follows:

In Section 2, we prove the characterization result for Azéma's supermartingales. To state it, we need to define a special class of submartingales, whose definition goes back to Yor [31], and which was also studied in [17] (under more general conditions):

Definition 1.2. Let (X_t) be a positive local submartingale, which decomposes as:

$$X_t = N_t + A_t.$$

We say that (X_t) is of class (Σ) if:

(1) (N_t) is a continuous local martingale, with $N_0 = 0$;
(2) (A_t) is a continuous increasing process, with $A_0 = 0$;
(3) the measure (dA_t) is carried by the set $\{t : X_t = 0\}$.

If additionally, (X_t) is of class (D), we shall say that (X_t) is of class (ΣD).

Now, consider the Doob-Meyer decomposition of Z_t^L:

$$Z_t^L \equiv 1 + \mu_t^L - A_t^L.$$

We prove that if (Z_t) is a nonnegative supermartingale, with $Z_0 = 0$, then, Z may be represented as $\mathbb{P}(L > t|\mathcal{F}_t)$, for some honest time L which avoids stopping times, if and only if $(X_t \equiv 1 - Z_t)$ is a submartingale of the class (Σ), with the limit condition:

$$\lim_{t\to\infty} X_t = 1.$$

We then apply our characterization to give many examples of enlargement formulae. We also extend Doob's maximal identity for continuous and nonnegative local martingales and some related results about the last time the maximum of such local martingales is reached.
In Section 3, we apply the results of Section 2 to obtain an analogue of Williams' path decomposition result for the supermartingale $\left(Z_t^L\right)$. We also establish some path decomposition results for certain classes of diffusion processes which play an important role in applications. In particular, we shall see that the pseudo-stopping times, introduced in [19], play an important role in path decompositions.
Eventually, in Section 4, we study the problem of initial enlargement of filtrations with A_∞^L to recover the progressive enlargement formula with L. This result was discovered by Jeulin ([13], p.58), but the proof given here is different and general enough to be adapted to handle with other situations where the powerful results of Jacod about initial enlargement do not apply.

2. A characterization of Azéma's supermartingale and applications

2.1. **The characterization of Azéma's supermartingale for honest times.** Azéma has studied in depth the supermartingale $Z_t^L = \mathbb{P}(L > t|\mathcal{F}_t)$ associated with an honest time L and has proved many of its interesting properties. A classical example of such a random time, which has received much attention in the literature ([13, 33], see [18] for a one parameter extension), is:

$$L = \sup\{t \leq 1 : \ B_t = 0\},$$

where as usual (B_t) denotes the standard Brownian Motion.

Let us briefly recall the results of Azéma. We assume that the conditions (**CA**) hold. We consider the Doob-Meyer decomposition of Z^L:

$$Z_t^L = 1 + \mu_t^L - A_t^L \tag{2.1}$$

The process $\left(A_t^L\right)$, which we shall sometimes note (A_t) in the sequel, is the dual predictable projection of the increasing process $\mathbf{1}_{\{L\leq t\}}$, and

$$\mu_t^L = \mathbb{E}\left(A_\infty^L \mid \mathcal{F}_t\right) - 1$$

Proposition 2.1 (Azéma [1])**.** *Let L be an honest time; then*

$$L = \sup\{t : \ Z_t = 1\},$$

and the measure dA_t is carried by the set $\{t : \ Z_t = 1\}$. In particular, A does not increase after L, i.e. $A_L = A_\infty$.

It follows from this proposition that

$$X_t \equiv 1 - Z_t^L$$

is of the class (ΣD), with: $X_0 = 0$ and $\lim_{t\to\infty} X_t = 1$. It also follows easily from Proposition 2.1 (see [1] or apply Corollary 4.7 in [17]) that A_∞ is distributed with the standard exponential law.

To prove our main theorem, we shall need the following very useful lemma, which also appears in the papers of Azéma, Meyer and Yor [4] and Azéma and Yor [7], in a more general framework (they consider ends of bounded optional right closed sets)[1]. Here, with our assumptions, the proof is particulary simple.

Lemma 2.2. *Let (X_t) be a submartingale of the class (ΣD) and let*

$$L = \sup\{t: \ X_t = 0\}.$$

Assume further that:

$$\mathbb{P}(X_\infty = 0) = 0.$$

Then:

$$X_t = \mathbb{E}\left(X_\infty \mathbf{1}_{\{L \leq t\}} | \mathcal{F}_t\right). \tag{2.2}$$

Proof. Since (X_t) is continuous, the set $\{t: \ X_t = 0\}$ is a predictable closed set. Let us remark that:

$$X_\infty \mathbf{1}_{\{L\leq t\}} = X_{d_t},$$

where

$$d_t \equiv \inf\{s > t: \ X_s = 0\}.$$

Hence, we have:

$$\begin{aligned}\mathbb{E}\left(X_\infty \mathbf{1}_{\{L\leq t\}}|\mathcal{F}_t\right) &= \mathbb{E}\left(X_{d_t}|\mathcal{F}_t\right) \\ &= \mathbb{E}\left(N_{d_t}|\mathcal{F}_t\right) + \mathbb{E}\left(A_{d_t}|\mathcal{F}_t\right).\end{aligned}$$

Now, from the optional stopping theorem, we have:

$$\mathbb{E}\left(N_{d_t}|\mathcal{F}_t\right) = N_t.$$

Moreover, as (dA_t) is carried by the set $\{t: \ X_t = 0\}$, we have:

$$A_{d_t} = A_t.$$

We can thus conclude that:

$$\mathbb{E}\left(X_\infty \mathbf{1}_{\{L\leq t\}}|\mathcal{F}_t\right) = N_t + A_t = X_t,$$

and this completes the proof. $\square$

[1] The results in [4] are probably the very last legacy of Paul André Meyer to the general theory of stochastic processes.

Corollary 2.3. *With the notations and assumptions of Lemma 2.2, (A_t) is the dual predictable projection of the raw increasing process $(X_\infty \mathbf{1}_{L\leq t})$, i.e. for any nonnegative or bounded predictable process (h_t),*

$$\mathbb{E}\left(X_\infty h_L\right) = \mathbb{E}\left(\int_0^\infty h_u dA_u\right).$$

Now, we can state our characterization theorem.

Theorem 2.4. *Let (X_t) be a submartingale of the class (ΣD) satisfying: $\lim_{t\to\infty} X_t = 1$. Let*

$$L = \sup\left\{t:\ X_t = 0\right\}.$$

Then (X_t) is related to the Azéma's supermartingale associated with L in the following way:

$$X_t = 1 - Z_t^L = \mathbb{P}\left(L \leq t | \mathcal{F}_t\right).$$

Consequently, if (Z_t) is a nonnegative supermartingale, with $Z_0 = 1$, then, Z may be represented as $\mathbf{P}\left(L > t|\mathcal{F}_t\right)$, for some honest time L which avoids stopping times, if and only if $(X_t \equiv 1 - Z_t)$ is a submartingale of the class (Σ), with the limit condition:

$$\lim_{t\to\infty} X_t = 1.$$

Proof. This is an immediate application of Lemma 2.2, with $X_\infty = 1$ and Proposition 2.1. □

2.2. **Last zero and maxima of nonnegative and continuous local martingales.** In [17], we computed the law of A_∞ when X is of the class (ΣD); in particular, we were interested in stopping times of the form

$$T \equiv \inf\left\{t:\ \varphi\left(A_t\right) X_t \geq 1\right\}, \tag{2.3}$$

for certain nonnegative Borel functions. In particular, we observed in [17] that $(\varphi\left(A_t\right) X_t)$ is of the class (Σ), and may be represented as:

$$\varphi\left(A_t\right) X_t = \int_0^t \varphi\left(A_u\right) dN_u + \Phi\left(A_t\right), \tag{2.4}$$

where $\Phi\left(x\right) = \int_0^x dz \varphi\left(z\right)$ (since X is continuous, this last formula can also be easily deduced from standard balayage arguments as exposed in [25], chapter VI). We also showed that if φ is a nonnegative locally bounded Borel function, such that $\int_0^\infty dz\varphi\left(z\right) = \infty$, then the stopping time T is almost surely finished:

$$\int_0^\infty dz\varphi\left(z\right) = \infty \Rightarrow T < \infty. \tag{2.5}$$

Here, we shall develop further the study of X on $[0, T]$, using Lemma 2.2 and Proposition 2.1. More precisely, we shall look for the Azéma's supermartingale and the dual predictable projection of the honest time

$$g_T \equiv \sup\left\{t < T:\ X_t = 0\right\}.$$

Proposition 2.5. *Let φ be a nonnegative locally bounded Borel function, such that $\varphi(x) > 0$, $\forall x > 0$ and such that $\int_0^\infty dz\varphi(z) = \infty$ and let X be a local submartingale of the class (Σ), such that $A_\infty = \infty$. Let T be defined as in (2.3) and let*

$$g_T \equiv \sup\{t < T: \ X_t = 0\}.$$

Then,

$$\mathbb{P}(g_T \leq t|\mathcal{F}_t) = \varphi(A_{t\wedge T}) X_{t\wedge T} = \int_0^{t\wedge T} \varphi(A_u)\, dN_u + \Phi(A_{t\wedge T}), \tag{2.6}$$

and g_T avoids all $(\mathcal{F}_t)$ stopping times. Consequently,

$$\mathbb{P}(A_T > x) = \exp\left(-\int_0^x dz\varphi(z)\right).$$

Proof. From (2.5), $T < \infty$, and consequently, $\varphi(A_T) X_T = 1$. Now, since $(\varphi(A_{t\wedge T}) X_{t\wedge T})$ is of the class (Σ), we can apply Lemma 2.2 (with $X_\infty = 1$) to obtain the identity (2.6). From (2.4), we also have: $\mathbb{P}(g_T \leq t|\mathcal{F}_t) = \int_0^{t\wedge T} \varphi(A_u)\, dN_u + \Phi(A_{t\wedge T})$, and hence the dual predictable projection of $(\mathbf{1}_{g_T \leq t})$ is $(\Phi(A_{t\wedge T}))$. Now, since all $(\mathcal{F}_t)$ martingales are continuous and $(\Phi(A_{t\wedge T}))$ is continuous, all $(\mathcal{F}_t)$ stopping times are predictable, and consequently, g_T avoids all $(\mathcal{F}_t)$ stopping times. Indeed, for any $(\mathcal{F}_t)$ stopping time R,

$$\mathbf{E}\left[\mathbf{1}_{\{g_T = R\}}\right] = \mathbb{E}\left[(\Delta\Phi(A_{R\wedge T}))\right] = 0.$$

Thus we get $P(g_T = R) = 0$.

The fact that $\mathbb{P}(A_T > x) = \exp\left(-\int_0^x dz\varphi(z)\right)$ is a consequence of the fact that $\Phi(A_T)$ is distributed as a random variable with the standard exponential law (see Proposition 2.1). □

It may happen that $T = \infty$, or in other words, $X_\infty = \varphi(A_\infty)$, for some nonnegative Borel functions, in which case we have the following proposition:

Proposition 2.6. *Let φ be a nonnegative Borel function such that $1/\varphi$ is locally bounded, and let X be a submartingale of the class (ΣD), such that $X_\infty = \varphi(A_\infty)$. Define:*

$$g \equiv \sup\{t: \ X_t = 0\}.$$

If $\varphi(A_\infty) > 0$, a.s., then:

$$\mathbb{P}(g \leq t|\mathcal{F}_t) = \frac{X_t}{\varphi(A_t)} = \int_0^t \frac{dN_u}{\varphi(A_u)} + \int_0^{A_t} \frac{dz}{\varphi(z)}. \tag{2.7}$$

Consequently,

$$\mathbb{P}(A_\infty > x) = \exp\left(-\int_0^x \frac{dz}{\varphi(z)}\right).$$

Proof. From Lemma 2.2, we have:

$$X_t = \mathbb{E}\left(X_\infty \mathbf{1}_{\{g \leq t\}}|\mathcal{F}_t\right) = \mathbb{E}\left(\varphi(A_\infty)\mathbf{1}_{\{g \leq t\}}|\mathcal{F}_t\right).$$

But from Proposition 2.1, we also have:

$$\varphi\left(A_{\infty}\right)\mathbf{1}_{\{g\leq t\}}=\varphi\left(A_{t}\right)\mathbf{1}_{\{g\leq t\}},$$

and consequently,

$$X_{t}=\varphi\left(A_{t}\right)\mathbb{P}\left(g\leq t|\mathcal{F}_{t}\right),$$

and (2.7) follows easily from (2.4). Eventually, the law of A_{∞} follows from the fact that the dual predictable projection of $\mathbf{1}_{\{g\leq t\}}$, which is $\int_{0}^{A_{t}}\frac{dz}{\varphi(z)}$, taken at $t=\infty$, follows the standard exponential law. □

Now, we give some applications of Propositions 2.5 and 2.6 to the computation of the law of the maxima of nonnegative continuous local martingales. So far, in the literature, only the cases of continuous uniformly integrable martingales or continuous and convergent martingales vanishing at 0 have been dealt with ([26, 28]), except the notable case when the nonnegative local martingale converges to zero (see [20] for more details and some extensions to some discontinuous local martingales). Moreover, nothing is said about the law of the last time this maximum is reached. Here, we shall deal with the case of a nonnegative continuous local martingale (M_t). Without loss of generality, we can assume that $M_0=1$. With the notation

$$\overline{M}_{t}\equiv\sup_{u\leq t}M_{u},$$

we have:

Corollary 2.7. *Let (M_t) be a continuous nonnegative local martingale starting from 1, and such that $M_{\infty}=\psi\left(\overline{M}_{\infty}\right)$, with ψ a nonnegative Borel function such that $\psi\left(x\right)<x$, $\forall x\geq 1$ and $1/(x-\psi\left(x\right))$ is locally bounded. Define*

$$g\equiv\sup\left\{t:\ M_{t}=\overline{M}_{t}\right\}.$$

Then,

$$\begin{aligned}\mathbb{P}\left(g\leq t|\mathcal{F}_{t}\right) &= \frac{\overline{M}_{t}-M_{t}}{\overline{M}_{t}-\psi\left(\overline{M}_{t}\right)}\\ &= -\int_{0}^{t}\frac{dM_{u}}{\overline{M}_{u}-\psi\left(\overline{M}_{u}\right)}+\int_{1}^{\overline{M}_{t}}\frac{dz}{z-\psi\left(z\right)}.\end{aligned}$$

Hence, the dual predictable projection of the raw increasing process $\left(\mathbf{1}_{\{g\leq t\}}\right)$, is $\left(\int_{0}^{\overline{M}_{t}}\frac{dz}{z-\psi(z)}\right)$. Consequently, we have:

$$\mathbb{P}\left(\overline{M}_{\infty}>x\right)=\exp\left(-\int_{1}^{x}\frac{dz}{z-\psi\left(z\right)}\right),\ \forall x\geq 1. \tag{2.8}$$

In particular, when $\psi \equiv 0$, i.e. when $M_\infty = 0$, we recover the following results which appears in [20]*:*

$$\begin{aligned} \mathbb{P}(g > t | \mathcal{F}_t) &= \frac{M_t}{\overline{M}_t}, \\ \mathbb{P}\left(\overline{M}_\infty > x\right) = &= \frac{1}{x}, \ \forall x \geq 1. \end{aligned}$$

Proof. Let us define

$$X_t \equiv 1 - \frac{M_t}{\overline{M}_t}.$$

An application of Itô's formula, combined with the fact that $d\overline{M}_t$ is carried by the set $\left\{t : \ M_t = \overline{M}_t\right\}$, yields:

$$X_t = -\int_0^t \frac{dM_u}{\overline{M}_u} + \log\left(\overline{M}_t\right),$$

and since $\{t : \ X_t = 0\} = \left\{t : \ M_t = \overline{M}_t\right\}$, we easily deduce that X (which is bounded by 1) is of the class (ΣD). With the notations of Proposition 2.7, we have $A_t = \log\left(\overline{M}_t\right)$ and $\varphi(x) \equiv 1 - \dfrac{\psi(\exp(x))}{\exp(x)}$. Since $\psi(x) < x$, we have $\varphi(A_\infty) > 0$, and the results of the corollary follow from an application of Proposition 2.6 and some elementary calculations. □

Remark 2.8. The situation described in the previous Corollary often occurs in the resolution of the Skorokhod stopping problem: the Azéma-Yor solution to Skorokhod's embedding problem relies upon the construction of a Brownian martingale $M_t = B_{t \wedge T}$ such that $M_T = \psi(\overline{M}_T)$.

Furthermore, any local martingale (Y_t) of the filtration $(\mathcal{F}_t)$ is a semimartingale in the filtration $(\mathcal{F}_t^g)$ and decomposes as:

$$Y_t = \widetilde{Y}_t + \int_0^{t \wedge g} \frac{d\langle Y, M\rangle_u}{M_u - \psi\left(\overline{M}_u\right)} - \int_g^t \frac{d\langle Y, M\rangle_u}{M_g - M_u}.$$

Remark 2.9. For example, assume that $\psi(x) \equiv x - 1$. Then we get:

$$\mathbb{P}(g \leq t | \mathcal{F}_t) = \overline{M}_t - M_t,$$

and

$$\mathbb{P}\left(\overline{M}_\infty > x\right) = \exp\left(-(x-1)\right).$$

We can say a little more about the honest time g:

Proposition 2.10. *In fact, g is the unique time M reaches its global maximum, i.e.*

$$g = \inf\left\{t : \ \overline{M}_t = \overline{M}_\infty\right\}.$$

Proof. Indeed, the measure $d\overline{M}_t$ is carried by the set $\left\{t : \ M_t = \overline{M}_t\right\}$. As $g = \sup\left\{t : \ M_t = \overline{M}_t\right\}$, the process $\left(\overline{M}_t\right)$ does not grow after g, which also satisfies:

$$g = \inf\left\{t : \ \overline{M}_t = \overline{M}_\infty\right\}.$$

□

2.3. **Some fundamental examples.** In the sequel, we give some explicit (yet generic) computations of Azéma's supermartingales for some honest times associated with some very well known stochastic processes. These computations are the first steps towards the path decompositions proved in the next section.

2.3.1. *A Brownian example.* First, consider (B_t), the standard Brownian Motion, and let $T_1 = \inf\{t \geq 0 : B_t = 1)$. Let $\sigma = \sup\{t < T_1 : B_t = 0\}$. Then $B^+_{t\wedge T_1}$ satisfies the conditions of Theorem 2.4, and hence:

$$\mathbb{P}(\sigma \leq t|\mathcal{F}_t) = B^+_{t\wedge T_1} = \int_0^{t\wedge T_1} \mathbf{1}_{B_u>0}\, dB_u + \frac{1}{2}\ell_{t\wedge T_1},$$

where (ℓ_t) is the local time of B at 0. This example plays an important role in the celebrated Williams' path decomposition for the standard Brownian Motion on $[0, T_1]$. This result is usually obtained by exploiting the strong Markov property of the Brownian Motion. Our method shows that we can get rid of the Markov property, and this will allow us to get similar formulae in the more general context of continuous local martingales, as is shown in the next paragraph.

One can also consider $T_{\pm 1} = \inf\{t \geq 0 : |B_t| = 1)$ and $\tau = \sup\{t < T_{\pm 1} : |B_t| = 0\}$. $|B_{t\wedge T_{\pm 1}}|$ satisfies the conditions of Theorem 2.4, and hence:

$$\mathbb{P}(\tau \leq t|\mathcal{F}_t) = |B_{t\wedge T_{\pm 1}}| = \int_0^{t\wedge T_{\pm 1}} sgn\,(B_u)\, dB_u + \ell_{t\wedge T_{\pm 1}}.$$

2.3.2. *Generalization to continuous local martingales.* More generally, consider a continuous local martingale (M_t) such that $M_0 = 0$ and $< M >_\infty = \infty$, a.s.; let $T_1 = \inf\{t \geq 0 : M_t = 1\}$ and $\sigma = \sup\{t < T_1 : M_t = 0\}$. Then, again, an application of Theorem 2.4 gives:

$$\mathbb{P}(\sigma \leq t|\mathcal{F}_t) = M^+_{t\wedge T_1} = \int_0^{t\wedge T_1} \mathbf{1}_{M_u>0}\, dM_u + \frac{1}{2}L_{t\wedge T_1},$$

where (L_t) is the local time of M at 0.

2.3.3. *Recurrent diffusions.* Let (Y_t) be a real continuous recurrent diffusion process, with $Y_0 = 0$. Then from the general theory of diffusion processes, there exists a unique continuous and strictly increasing function s, with $s(0) = 0$, $\lim_{x\to+\infty} s(x) = +\infty$, $\lim_{x\to-\infty} s(x) = -\infty$, such that $s(Y_t)$ is a continuous local martingale. Our aim is to establish some results analogue to those established for the Brownian Motion and recurrent continuous local martingales. Let

$$T_1 \equiv \inf\{t \geq 0 : Y_t = 1).$$

Now, if we define

$$X_t \equiv \frac{s(Y_{t\wedge T_1})^+}{s(1)},$$

we easily note that X is a local submartingale of the class (Σ) which satisfies the hypotheses of Theorem 2.4. Consequently, if we note

$$\sigma = \sup \{t < T_1 : \ Y_t = 0\},$$

we have:

$$\mathbb{P}(\sigma \leq t | \mathcal{F}_t) = \frac{s(Y_{t \wedge T_1})^+}{s(1)}.$$

2.3.4. *Nonnegative continuous martingales which vanish at infinity.* Now let (M_t) be a positive local martingale, such that: $M_0 = x$, $x > 0$ and $\lim_{t \to \infty} M_t = 0$. Then, Tanaka's formula shows us that $\left(1 - \frac{M_t}{y} \wedge 1\right)$, for $0 \leq y \leq x$, is a local submartingale of the class (Σ) satisfying the assumptions of Theorem 2.4, and hence with

$$g = \sup \{t : \ M_t = y\},$$

we have:

$$\mathbb{P}(g > t | \mathcal{F}_t) = \frac{M_t}{y} \wedge 1 = 1 + \frac{1}{y} \int_0^t \mathbf{1}_{(M_u < y)} \, dM_u - \frac{1}{2y} L_t^y,$$

where (L_t^y) is the local time of M at y.

2.3.5. *Transient diffusions.* As an illustration of the previous example, consider (R_t), a transient diffusion with values in $[0, \infty)$, which has $\{0\}$ as entrance boundary. Let s be a scale function for R, which we can choose such that:

$$s(0) = -\infty, \text{ and } s(\infty) = 0.$$

Then, under the law $\mathbb{P}_x$, for any $x > 0$, the local martingale $(M_t = -s(R_t))$ satisfies the conditions of the previous example and for $0 \leq x \leq y$, we have:

$$\mathbb{P}_x(g_y > t | \mathcal{F}_t) = \frac{s(R_t)}{s(y)} \wedge 1 = 1 + \frac{1}{s(y)} \int_0^t \mathbf{1}_{(R_u > y)} \, d(s(R_u)) + \frac{1}{2s(y)} L_t^{s(y)}, \tag{2.9}$$

where $\left(L_t^{s(y)}\right)$ is the local time of $s(R)$ at $s(y)$, and where

$$g_y = \sup \{t : \ R_t = y\}.$$

Formula (2.9) was the key point to derive the distribution of g_y in [23], Theorem 6.1, p.326.

3. Path decompositions

In this section, inspired by Williams' path decompositions for the standard Brownian Motion and for transient diffusions given their minima, we establish path decompositions for Azéma's supermartingales and some families of recurrent and transient diffusions. What follows is similar in spirit to what we have done in [20], but in an additive setting, and of course the results are different. It is also an opportunity to show that the techniques of progressive and initial enlargements of filtrations can be combined to prove, quite shortly, some non trivial path decomposition results.

Let us recall briefly the random times introduced by D. Williams to study the paths of a standard Brownian Motion B:

$$T_1 = \inf\{t : \ B_t = 1\}, \ \sigma = \sup\{t < T_1 : \ B_t = 0\};$$

and

$$\rho = \sup\{u < \sigma : \ B_u = S_u\}, \text{ where } S_u = \sup_{s \leq u} B_s.$$

D. Wiliams ([30]) discovered the remarkable fact that although ρ is not a stopping time, it satisfies nevertheless the optional stopping theorem, i.e. for every bounded martingale (M_t) of the filtration $(\mathcal{F}_t)$, we have:

$$\mathbb{E}M_\rho = \mathbb{E}M_\infty.$$

In [19], we have called such random times pseudo-stopping times and we have characterized them. Before stating and proving our results, we shall first recall in the next subsection some standard facts about pseudo-stopping times and multiple enlargements of filtrations.

3.1. **Basic facts about pseudo-stopping times and double enlargements of filtrations.**

3.1.1. *Pseudo-stopping times.* In [19], following D. Williams, we have proposed the following generalization of stopping times:

Definition 3.1 ([19])**.** Let $\rho : \ (\Omega, \mathcal{F}) \to \mathbf{R}_+$ be a random time; ρ is called a pseudo-stopping time if for every bounded $(\mathcal{F}_t)$ martingale we have:

$$\mathbf{E}(M_\rho) = \mathbf{E}(M_0).$$

David Williams ([30]) gave the first example of such a random time and the following systematic construction is established in [19]:

Proposition 3.2 ([19])**.** *Let L be an honest time. Then, under the conditions **(CA)**,*

$$\rho \equiv \sup\left\{t < L : \ Z_t^L = \inf_{u \leq L} Z_u^L\right\},$$

is a pseudo-stopping time, with

$$Z_t^\rho \equiv \mathbf{P}(\rho > t \mid \mathcal{F}_t) = \inf_{u \leq t} Z_u^L,$$

and Z_ρ^ρ follows the uniform distribution on $(0,1)$.

The following property, also proved in [19], is essential in studying path decompositions:

Proposition 3.3 ([19])**.** *Let ρ be a pseudo-stopping time and let M_t be an $(\mathcal{F}_t)$ local martingale. Then $(M_{t\wedge\rho})$ is an $(\mathcal{F}_t^\rho)$ local martingale.*

To conclude, let us illustrate Proposition 3.2 with an example. Let Y be a recurrent diffusion; with the notations and assumptions of paragraph 2.3.3,

$$\rho \equiv \sup\left\{ t < \sigma : \ Y_t = \max_{u \leq \sigma} Y_u \right\},$$

is a pseudo-stopping time.

3.1.2. *Double enlargements of filtrations.* We recall some not so well known results of Jeulin ([13]) about successive progressive enlargements of filtrations. The reader can also refer to [11] for a more recent exposition (summary) of these facts.

Proposition 3.4 (Jeulin [13])**.** *Let L be an honest time for the filtration $(\mathcal{F}_t)$ and let ρ be an honest time for $\left(\mathcal{F}_t^L\right)$, and define $\left(\mathcal{F}_t^{L,\rho}\right)$ the filtration obtained by enlarging progressively $\left(\mathcal{F}_t^L\right)$ with ρ. Then, any $(\mathcal{F}_t)$ local martingale (M_t) is a semimartingale in $\left(\mathcal{F}_t^{L,\rho}\right)$ and decomposes as:*

$$M_t = \widetilde{M}_t + \int_0^{t\wedge\rho} \frac{d\langle M, Z^\rho\rangle_u}{Z^\rho_{u-}} + \int_\rho^{t\wedge L} \frac{d\langle M, Z^L - Z^\rho\rangle_u}{Z^L_{u-} - Z^\rho_{u-}} - \int_L^t \frac{d\langle M, Z^L\rangle_u}{1 - Z^L_{u-}}, \quad (3.1)$$

where $\left(\widetilde{M}_t\right)$ is an $\left(\mathcal{F}_t^{L,\rho}\right)$ local martingale.

Honest times enjoy the remarkable property that every $(\mathcal{F}_t)$ semimartingale is an $\left(\mathcal{F}_t^L\right)$ semimartingale, or in the jargon of the theory of enlargements of filtrations, the pair of filtrations $\left((\mathcal{F}_t), \left(\mathcal{F}_t^L\right)\right)$ satisfy the $(\mathcal{H}')$ hypothesis. The previous proposition shows that there might be non-honest times which enjoy this property; indeed, the pseudo-stopping times defined in Proposition 3.2 have this property:

Corollary 3.5. *Let us consider the pseudo-stopping time defined in Proposition 3.2. Then, every $(\mathcal{F}_t)$ semimartingale is an $(\mathcal{F}_t^\rho)$ semimartingale, or in other words, the pair of filtrations $((\mathcal{F}_t), (\mathcal{F}_t^\rho))$ satisfy the $(\mathcal{H}')$ hypothesis.*

Proof. It suffices to prove that every $((\mathcal{F}_t))$ local martingale (M_t) is an $(\mathcal{F}_t^\rho)$ semimartingale. From Proposition 3.4, every $((\mathcal{F}_t))$ local martingale is an $\left(\mathcal{F}_t^{L,\rho}\right)$ semimartingale, and since $\mathcal{F}_t^\rho \subset \mathcal{F}_t^{L,\rho}$ and (M_t) is $(\mathcal{F}_t^\rho)$ adapted, it follows from a well known result of Stricker (see [10], [24]) that (M_t) is also an $(\mathcal{F}_t^\rho)$ semimartingale. $\square$

We shall also need another result of Jeulin which certainly deserves more attention than what it has actually received: the problem of initial enlargement with A_∞^L (we shall give an elementary proof of this result in the next section).

Proposition 3.6. *Let T be a totally inaccessible stopping time, such that $\mathbb{P}(T > 0) = 1$ and let (A_t) be the $(\mathcal{F}_t)$ dual predictable projection of $(\mathbf{1}_{T \leq t})$. Then the following hold:*

(1) *A is continuous and* $T = \inf\{t : A_t = A_T\} = \sup\{t : A_t = A_T\}$ *(Azéma [1]);*
(2) *define* $\mathcal{G}_t \equiv \bigcap_{\varepsilon>0}(\mathcal{F}_{t+\varepsilon} \vee \sigma(A_T))$*; then every continuous* $(\mathcal{F}_t)$ *martingale is a* $(\mathcal{G}_t)$ *martingale (Jeulin [13]).*

Remark 3.7. In fact, every $(\mathcal{F}_t)$ martingale, which does not jump at T, is a $(\mathcal{G}_t)$ martingale.

Corollary 3.8. (1) *If L is an honest time which avoids stopping times, then every* $(\mathcal{F}_t)$ *martingale is an* $\left(\mathcal{F}_t^{L,\sigma(A_\infty^L)}\right)$ *semimartingale and the decomposition formula is the same as the decomposition formula (1.2):*

$$M_t = \widetilde{M}_t + \int_0^{t\wedge L} \frac{d\langle M, Z^L\rangle_s}{Z_{s_-}^L} + \int_L^t \frac{d\langle M, Z^L\rangle_s}{1 - Z_{s_-}^L}.$$

(2) *Similarly, under the assumptions* (**CA**)*, the pseudo-stopping times of Proposition 3.2 are inaccessible stopping times for the filtration* $\left(\mathcal{F}_t^{L,\rho}\right)$ *and* $\left(\log\left(\frac{1}{Z_{t\wedge\rho}^\rho}\right)\right)$ *is the* $\left(\mathcal{F}_t^{L,\rho}\right)$ *dual predictable projection of* $(\mathbf{1}_{\rho\leq t})$*. Here again, every* $(\mathcal{F}_t)$ *martingale is an* $\left(\mathcal{F}_t^{L,\sigma(Z_\rho^\rho)}\right)$ *semimartingale whose decomposition is given by (3.1):*

$$M_t = \widetilde{M}_t + \int_\rho^{t\wedge L} \frac{d\langle M, Z^L\rangle_u}{Z_\rho^L - Z_u^\rho} - \int_L^t \frac{d\langle M, Z^L\rangle_u}{1 - Z_u^L}, \tag{3.2}$$

where $\left(\widetilde{M}_t\right)$ *is an* $\left(\mathcal{F}_t^{L,\rho}\right)$ *and* $\left(\mathcal{F}_t^{L,\sigma(Z_\rho^\rho)}\right)$ *local martingale, every continuous* $\left(\mathcal{F}_t^{L,\rho}\right)$ *martingale being an* $\left(\mathcal{F}_t^{L,\sigma(Z_\rho^\rho)}\right)$ *martingale.*

Proof. (1). This first point follows from the fact that L is a totaly inaccessible stopping time for $\left(\mathcal{F}_t^L\right)$ (see [14]), and Proposition 3.6 can be applied with $A_T \equiv A_\infty^L$.

(2). First, we note from Proposition 3.2 that (Z_t^ρ) is a continuous and decreasing process ($Z_t^\rho = 1 - A_t^\rho$). Moreover, from a result of Jeulin and Yor ([14]), $\mathbf{1}_{(\rho\leq t)} - \int_0^{t\wedge\rho} \frac{A_u^\rho}{Z_u^\rho} = \mathbf{1}_{(\rho\leq t)} - \log\left(\frac{1}{Z_{t\wedge\rho}^\rho}\right)$ is an $(\mathcal{F}_t^\rho)$ martingale. It remains a martingale in $\left(\mathcal{F}_t^{L,\rho}\right)$, since it is of finite variation and $\rho < L$. Consequently, $\left(\log\left(\frac{1}{Z_{t\wedge\rho}^\rho}\right)\right)$ is also the $\left(\mathcal{F}_t^{L,\rho}\right)$ dual predictable projection of $(\mathbf{1}_{\rho\leq t})$ and the announced results now easily follow from Propositions 3.4 and 3.6. □

3.2. **An analogue of Williams' path decomposition theorem for** $\mathbb{P}(\mathbf{L} \leq \mathbf{t}|\mathcal{F}_\mathbf{t})$**.** We are going to use techniques from both stochastic calculus and the general theory of stochastic processes (the Dubins-Schwarz theorem and the decomposition formula in the larger filtration) to generalize some fragments of Williams' path decomposition for the standard Brownian to more general processes, namely the submartingale $(\mathbb{P}(L \leq t|\mathcal{F}_t))$, associated with an honest time L, under the conditions (**CA**).

Let (X_t) be a submartingale of the class (Σ) satisfying:

$$\lim_{t\to\infty} X_t = 1. \tag{3.3}$$

Let

$$L \equiv \sup\{t: \ X_t = 0\}.$$

Recall (Theorem 2.4) that

$$X_t = 1 - Z_t^L = \mathbf{P}(L \leq t|\mathcal{F}_t).$$

Let us also define the random time:

$$\rho = \sup\{t < L: \ X_t = S_L\},$$

where

$$S_t = \sup_{u\leq t} X_u.$$

Theorem 3.9. *Let*

$$X_t = N_t + A_t,$$

be a submartingale of the class (Σ) *satisfying (3.3), and let* $L = \sup\{t: \ X_t = 0\}$. *Then:*

1. *the process* (X_t) *is, up to the time change* $(\langle N\rangle_t)$*, a reflected Brownian Motion started from* 0*, stopped when it first hits* 1*.*
2. *The random time* ρ *is a pseudo-stopping time and*

   $$\mathbb{P}(\rho > t|\mathcal{F}_t) = 1 - S_t.$$

   Moreover, X_ρ *is uniformly distributed on* $(0,1)$*, and conditionally on* $X_\rho = m$*,* (X_t) *is up to the time change* $(\langle N\rangle_t)$*, a reflected Brownian Motion started from* 0 *and stopped when it first hits* m*.*
3. *Conditionally on* $\mathcal{F}_L$*, the process* (X_{L+t}) *is, up to the time change* $(\langle N\rangle_{L+t} - \langle N\rangle_L)$*, a Bessel process of dimension* 3*, started from* 0*, and stopped when it first hits* 1*.*

Proof. (1). First, from Skorokhod's reflection lemma (see [25] or [17]), we have:

$$A_t = \sup_{u\leq t}(-N_u).$$

Moreover, there exists a Brownian Motion (β_u) such that:

$$N_t = \beta_{\langle N\rangle_t}.$$

Hence, up to the time change $(\langle N\rangle_t)$, (X_t) has the same decomposition as the absolute value of a Brownian Motion (this is immediate from Tanaka's formula). Thus it is a time changed reflected Brownian Motion.

(2). The first point follows immediately from Proposition 3.2: indeed, ρ is a pseudo-stopping time and X_ρ is equal to Z^ρ_ρ, which is uniformly distributed. The second point follows from a combination of Proposition 3.4 and Corollary 3.8. Indeed, in the filtration $\left(\mathcal{F}_t^{L,\sigma(X_\rho)}\right)$, obtained by initially enlarging the filtration $\left(\mathcal{F}_t^L\right)$ with $\log\left(\frac{1}{Z^\rho_{t\wedge\rho}}\right) = \log\left(\frac{1}{1-X_\rho}\right)$, we have:

$$X_{t\wedge\rho} = N_{t\wedge\rho} + A_{t\wedge\rho}.$$

(3). We first note that, since $X_L = 0$, we have $N_L = -A_L$, and consequently,

$$X_{L+t} = N_{L+t} - N_L.$$

Now, using the fact that $X_t = 1 - Z^L_t = \mathbf{P}\left(L \le t | \mathcal{F}_t\right)$ the decomposition formula (1.2) yields:

$$X_{L+t} = N_{L+t} - N_L = \widetilde{N}_t + \int_0^t \frac{d\langle N\rangle_{L+u}}{X_{L+u}},$$

where $\widetilde{N}$ is an $\left(\mathcal{F}_t^L\right)$ martingale. Now, the result follows from the fact that the Bessel process of dimension 3 (R_t) can be characterized as the unique solution to the stochastic differential equation:

$$dR_t = dB_t + \frac{dt}{R_t},$$

where (B_t) is a one dimensional Brownian Motion. □

As an illustration of the above theorem, let us consider

$$X_t \equiv \alpha B_t^+ + \beta B_t^-,$$

where B is the standard Brownian Motion and $\alpha > 0, \beta > 0$. Let $T_1 = \inf\{t :\ X_t = 1\}$. Then, it is easy to check that $(X_{t\wedge T_1})$ satisfies the assumptions of the Theorem 3.9 with the time change

$$\langle N\rangle_t = \alpha^2 \int_0^t \mathbf{1}_{(B_u>0)}\, du + \beta^2 \int_0^t \mathbf{1}_{(B_u\le 0)}\, du.$$

3.3. **Path decompositions for some recurrent diffusions.** D. Williams's path decomposition also admits a generalization to the wider class of recurrent diffusions (Y_t), satisfying the stochastic differential equation:

$$Y_t = B_t + \int_0^t b\left(Y_u\right) du, \tag{3.4}$$

where (B_t) is the standard Brownian Motion, and b is a Borel integrable function which allows existence and uniqueness for equation (3.4). We note $\mathcal{L}$ the infinitesimal generator of this diffusion:

$$\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2} + b\left(x\right)\frac{d}{dx}.$$

Let $T_1 \equiv \inf\{t : Y_t = 1)$, and denote by s the scale function of Y, which is strictly increasing and which vanishes at zero, i.e:

$$s(z) = \int_0^z \exp\left(-2\widehat{b}(y)\right) dy,$$

where

$$\widehat{b}(y) = \int_0^y b(u)\, du.$$

From the results of paragraph (2.3.3), if we define

$$\sigma = \sup\{t < T_1 : Y_t = 0\},$$

we have, with (L_t) the local time at 0 of the local martingale $s(Y)$:

$$\begin{aligned}
\mathbb{P}(\sigma \le t | \mathcal{F}_t) &= \frac{s(Y_{t\wedge T_1})^+}{s(1)} \\
&= \frac{1}{s(1)} \int_0^{t\wedge T_1} \mathbf{1}_{(Y_u>0)} s'(Y_u)\, dB_u + \frac{1}{2s(1)} L_{t\wedge T_1},
\end{aligned}$$

where the last equality is obtained by an application of Tanaka's formula. Moreover, from Proposition (3.2),

$$\rho \equiv \sup\left\{t < \sigma : Y_t = \max_{u\le\sigma} Y_u\right\}, \tag{3.5}$$

is a pseudo-stopping time.

Proposition 3.10. *Let (Y_t) be a diffusion process satisfying equation (3.4). Define:*

$$\overline{Y}_t = \max_{u\le t} Y_u.$$

Then:

- *The process $(Y_{\sigma+t},\ t \le T_1 - \sigma)$ is an $(\mathcal{F}_{\sigma+t})$ diffusion, starting from 0, considered up to the first time it hits 1, and is independent of $\mathcal{F}_\sigma$. Its infinitesimal generator is given by:*

$$\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2} + \left(b(x) + \frac{s'(x)}{s(x)}\right)\frac{d}{dx}.$$

- *The random time ρ is a pseudo-stopping time and satisfies:*

$$\mathbb{P}(\rho > t | \mathcal{F}_t) = 1 - \frac{s\left(\overline{Y}_{t\wedge T_1}\right)^+}{s(1)}.$$

  Moreover, $Y_\rho = \overline{Y}_\sigma$ follows the same law as $s^{-1}(s(1)\, U)$, where U is a random variable following the uniform law on $(0,1)$, and is independent of the whole process $(Y_{\sigma+t},\ t \le T_1 - \sigma)$.
- ***Conditionally on $\mathbf{Y}_\rho = \mathbf{m}$,***
  1. *the process $(Y_t;\ t \le \rho)$ is a diffusion process, considered up to T_m, the first time when it hits m, with the same infinitesimal generator as Y.*

(2) *the process* $(Y_{\rho+t};\ t \leq \sigma - \rho)$ *is a* $(\mathcal{F}_{\rho+t})$ *diffusion process, started from* m*, considered up to* T_0*, the first time when it hits* 0*, and is independent of* $(Y_t;\ t \leq \rho)$*; its infinitesimal generator is given by:*

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(b\left(x\right) + \mathbf{1}_{(x>0)}\frac{s'\left(x\right)}{s\left(x\right) - s\left(m\right)}\right)\frac{d}{dx}.$$

Proof. The proof is based on enlargements arguments. First, we note that from Proposition 3.2 $s\left(Y_\rho\right)$ is distributed as $s\left(1\right)U$, where U follows the uniform law on $(0,1)$.

Now, let us study the path of Y on $[\sigma, T_1]$. From formula (1.2), the Brownian Motion B is a semimartingale in the filtration $(\mathcal{F}_t^\sigma)$ and decomposes as:

$$B_t = \widetilde{B}_t + \int_0^{t\wedge\sigma} \frac{d < B, Z^\sigma >_u}{Z_u^\sigma} + \int_\sigma^{t\wedge T_1} \frac{d < B, 1 - Z^\sigma >_u}{1 - Z_u^\sigma},$$

where $\widetilde{B}$ is a $(\mathcal{F}_t^\sigma)$ Brownian Motion (indeed it is a continuous local martingale with bracket t). Consequently, the diffusion Y, which is an$(\mathcal{F}_t)$ semimartingale, remains a semimartingale in $(\mathcal{F}_t^\sigma)$ and its decomposition is given by:

$$Y_t = \widetilde{B}_t + \int_0^t b\left(Y_u\right)du - \int_0^{t\wedge\sigma} \mathbf{1}_{Y_u>0}\frac{s'\left(Y_u\right)}{s\left(1\right) - s\left(Y_u\right)}du + \int_\sigma^{t\wedge T_1} \frac{s'\left(Y_u\right)}{s\left(Y_u\right)}du.$$

Now, considering $Y_{\sigma+t} - Y_\sigma = Y_{\sigma+t}$, for $t \leq T_1 - \sigma$, we obtain:

$$Y_{\sigma+t} = \left(\widetilde{B}_{\sigma+t} - \widetilde{B}_\sigma\right) + \int_\sigma^{\sigma+t} b\left(Y_u\right)du + \int_\sigma^{(\sigma+t)\wedge T_1} \frac{s'\left(Y_u\right)}{s\left(Y_u\right)}du.$$

Now, using the fact that σ is a stopping time for the filtration $(\mathcal{F}_t^\sigma)$, we have that $\left(\widetilde{B}_{\sigma+t} - \widetilde{B}_\sigma\right)$, which we note $\left(\widehat{B}_t\right)$, is a Brownian Motion, which is independent of $\mathcal{F}_\sigma^\sigma \supset \mathcal{F}_\sigma$. Consequently, for $t \leq T_1 - \sigma$, we have:

$$Y_{\sigma+t} = \widehat{B}_t + \int_0^t b\left(Y_{\sigma+u}\right)du + \int_0^{t\wedge(T_1-\sigma)} \frac{s'\left(Y_{\sigma+u}\right)}{s\left(Y_{\sigma+u}\right)}du,$$

and the result announced for the path on $[\sigma, T_1]$ follows now easily.

Now, let us consider the path of Y on $[0, \rho]$, and $[\rho, \sigma]$. For this, we enlarge initially the filtration $(\mathcal{F}_t^\sigma)$ with the variable Y_ρ: according to Proposition 3.4 and Corollary 3.8, for $t \leq \sigma$, B decomposes in $\left(\mathcal{F}_t^{\sigma,\sigma(Y_\rho)}\right)$, which we note $\left(\mathcal{F}_t^{\sigma,Y_\rho}\right)$ for notational convenience, as:

$$B_t = \widetilde{B}_t - \int_\rho^{t\wedge\sigma} \mathbf{1}_{(Y_u>0)}\frac{s'\left(Y_u\right)}{s\left(Y_\rho\right) - s\left(Y_u\right)}du,$$

where $\widetilde{B}$ is an $\left(\mathcal{F}_t^{\sigma,Y_\rho}\right)$ Brownian Motion which is independent of Y_ρ. Consequently, for $t \leq \rho$, Y decomposes in $\left(\mathcal{F}_t^{\sigma,Y_\rho}\right)$ as:

$$Y_t = \widetilde{B}_t + \int_0^t b\left(Y_u\right) du, \text{ for } t \leq \rho, \tag{3.6}$$

and for $t \leq (\sigma - \rho)$, we have:

$$Y_{\rho+t} = Y_\rho + \left(\widetilde{B}_{\rho+t} - \widetilde{B}_\rho\right) + \int_\rho^{t\wedge(\sigma-\rho)} b\left(Y_u\right) du - \int_\rho^{t\wedge(\sigma-\rho)} \frac{s'\left(Y_u\right)}{s\left(Y_\rho\right) - s\left(Y_u\right)} du. \tag{3.7}$$

Now,

$$\widehat{B}_t \equiv \widetilde{B}_{\rho+t} - \widetilde{B}_\rho$$

is again a standard Brownian Motion, independent of Y_ρ, and hence, conditionally on $Y_\rho = m$, the process $(Y_{\rho+t};\ t \leq \sigma - \rho)$ satisfies:

$$Y_{\rho+t} = m + \widehat{B}_t + \int_0^{t\wedge(\sigma-\rho)} b\left(Y_{\rho+u}\right) du + \int_0^{t\wedge(\sigma-\rho)} \frac{s'\left(Y_{\rho+u}\right)}{s\left(Y_{\rho+u}\right) - s\left(m\right)} du.$$

The statement of the Proposition now follows from the last equality and equation (3.6). □

Remark 3.11. When $b \equiv 0$, we have $s\left(x\right) = x$, and we recover D. Williams'path decomposition for the standard Brownian Motion.

3.4. **Path decompositions for some transient diffusions.** Now, we consider a special subfamily of the transient diffusions of paragraph 2.3.5 which play an important role in the extension of Pitman's theorem (see [33], p.46). More precisely, let (R_t) be any transient diffusion which takes its values in $(0, \infty)$, and satisfies:

$$R_t = x + B_t + \int_0^t c\left(R_u\right) du, \ , x > 0 \ t \geq 0, \tag{3.8}$$

where $c : \mathbb{R}_+ \to \mathbb{R}$ allows uniqueness in law for this equation. Noting $T_0 = \inf\{t :\ R_t = 0\}$, we assume that $\mathbb{P}_x\left(T_0 < \infty\right) = 0$, so that a scale function s of R may be chosen to satisfy:

$$s\left(0\right) = -\infty; s\left(\infty\right) = 0;\ \frac{1}{2}s'' + cs' = 0.$$

We keep the notations of paragraph 2.3.5: for $0 \leq x \leq y$, and

$$g_y = \sup\left\{t :\ R_t = y\right\},$$

we have:

$$\begin{aligned}
\mathbb{P}_x\left(g_y > t|\mathcal{F}_t\right) &= \frac{s\left(R_t\right)}{s\left(y\right)} \wedge 1 \\
&= 1 + \frac{1}{s\left(y\right)} \int_0^t \mathbf{1}_{R_u > y}\ s'\left(R_u\right) dB_u + \frac{1}{2s\left(y\right)} L_t^{s(y)},
\end{aligned}$$

where $L_t^{s(y)}$ is the local time of $s(R)$ at $s(y)$.

From Proposition 3.2, the random time:

$$\rho = \sup\left\{ t < g_y : \ R_t = \sup_{u \leq g_y} R_u \right\},$$

is a pseudo-stopping time and $\mathbb{P}(\rho > t | \mathcal{F}_t) = \dfrac{s\left(\sup_{u \leq t} R_u\right)}{s(y)} \wedge 1$. Now, likewise Proposition 3.10, the following path decomposition holds for the diffusion R:

Proposition 3.12. *Let (R_t) be a diffusion process satisfying equation (3.8). Then:*

- *The process $\left(R_{g_y+t},\ t \geq 0\right)$ is an $\left(\mathcal{F}_{g_y+t}\right)$ diffusion, starting from y, and is independent of $\mathcal{F}_{g_y}$. Its infinitesimal generator is given by:*

$$\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2} + \left(c(x) + \frac{s'(x)}{s(x) - s(y)} \right)\frac{d}{dx}.$$

- *The random time ρ is a pseudo-stopping time and $R_\rho = \overline{R}_{g_y}$ follows the same law as $s^{-1}(s(y)U)$, where U follows the uniform law on $(0,1)$, and is independent of the whole process $\left(R_{g_y+t},\ t \geq 0\right)$.*
- ***Conditionally on $\mathbf{R}_\rho = \mathbf{m}$**,*
  (1) *the process $(R_t;\ t \leq \rho)$ is a diffusion process, considered up to T_m, the first time when it hits m, with the same infinitesimal generator as R.*
  (2) *the process $(R_{\rho+t};\ t \leq g_y - \rho)$ is a $(\mathcal{F}_{\rho+t})$ diffusion process, started from m, considered up to T_y, the first time when it hits y, and is independent of $(R_t;\ t \leq \rho)$; its infinitesimal generator is given by:*

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(c(x) + \mathbf{1}_{(x>y)}\frac{s'(x)}{s(x) - s(m)} \right)\frac{d}{dx}.$$

Proof. The proof follows exactly the same lines as the proof of Proposition 3.10 and so we will not give it. □

4. Initial enlargement with A_∞^L

In this section, we deal with the problem of initial enlargement of filtration with the variable A_∞^L (under the assumptions $(\mathbf{CA})$), for which the results of Jacod ([16]) do not apply. As we have already mentioned it, this problem was already solved by Jeulin, but our proof is different and simple. Moreover, it can be adapted to deal with other situations where the usual techniques do not apply.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space and let L be an honest time. We assume that the conditions $(\mathbf{CA})$ hold. In the sequel, we shall note A_t, Z_t, μ_t for A_t^L, Z_t^L, μ_t^L. Let us define the new filtration

$$\mathcal{F}_t^{\sigma(A_\infty)} \equiv \bigcap_{\varepsilon>0} \left(\mathcal{F}_{t+\varepsilon} \vee \sigma\left(A_\infty\right)\right),$$

which satisfies the usual assumptions. The new information $\sigma\left(A_\infty\right)$ is brought in at the origin of time.

We first need the conditional laws of A_∞ which were obtained under conditions (**CA**) in [2] and in a more general setting and by different methods in [21].

Proposition 4.1 ([21],[2]). *Let G be a Borel bounded function. Define:*

$$M_t^G \equiv \mathbf{E}\left(G\left(A_\infty\right)|\mathcal{F}_t\right).$$

Then,

$$M_t^G = F\left(A_t\right) - \left(F\left(A_t\right) - G\left(A_t\right)\right)\left(1 - Z_t\right),$$

where

$$F\left(x\right) = \exp\left(x\right) \int_x^\infty dy \exp\left(-y\right) G\left(y\right).$$

Moreover, $\left(M_t^G\right)$ has the following stochastic integral representation:

$$M_t^G = \mathbf{E}\left[G\left(A_\infty\right)\right] + \int_0^t \left(F - G\right)\left(A_u\right) d\mu_u.$$

Now, define, for G any Borel bounded function,

$$\lambda_t\left(G\right) \equiv M_t^G = F\left(A_t\right) - \left(F\left(A_t\right) - G\left(A_t\right)\right)\left(1 - Z_t^g\right).$$

From Proposition 4.1, we also have:

$$\begin{aligned}\lambda_t\left(G\right) &= \mathbf{E}\left[G\left(A_\infty\right)\right] + \int_0^t \left(F - G\right)\left(A_s\right) d\mu_s \\ &\equiv \mathbf{E}\left[G\left(A_\infty\right)\right] + \int_0^t \dot{\lambda}_s\left(G\right) d\mu_s.\end{aligned}$$

Hence we have:

$$\lambda_t\left(G\right) = \int \lambda_t\left(dx\right) G\left(x\right),$$

with

$$\lambda_t\left(dx\right) = \left(1 - Z_t\right)\delta_{A_t}\left(dx\right) + Z_t \exp\left(A_t\right) \mathbf{1}_{\left(A_t,\infty\right)}\left(x\right) \exp\left(-x\right) dx,$$

where δ_{A_t} denotes the Dirac mass at A_t. Similarly, we have:

$$\dot{\lambda}_t\left(G\right) = \int \dot{\lambda}_t\left(dx\right) G\left(x\right),$$

with:

$$\dot{\lambda}_t\left(dx\right) = -\delta_{A_t}\left(dx\right) + \exp\left(A_t\right) \mathbf{1}_{\left(A_t,\infty\right)}\left(x\right) \exp\left(-x\right) dx.$$

It then follows that:

$$\dot{\lambda}_t\left(dx\right) = \lambda_t\left(dx\right) \rho\left(x, t\right), \tag{4.1}$$

with

$$\rho(x,t) = \frac{1}{Z_t}\mathbf{1}_{\{x>A_t\}} - \frac{1}{1-Z_t}\mathbf{1}_{\{x=A_t\}}. \tag{4.2}$$

Now we can state our result about initial expansion with A_∞.

Theorem 4.2. *Let L be an honest time. We assume, as usual, that the conditions **(CA)** hold. Then, every $(\mathcal{F}_t)$ local martingale M is an $\left(\mathcal{F}_t^{\sigma(A_\infty)}\right)$ semimartingale and decomposes as:*

$$M_t = \widetilde{M}_t + \int_0^t \mathbf{1}_{\{L>s\}}\frac{d\langle M,\mu\rangle_s}{Z_s} - \int_0^t \mathbf{1}_{\{L\le s\}}\frac{d\langle M,\mu\rangle_s}{1-Z_s}, \tag{4.3}$$

where $\left(\widetilde{M}_t\right)_{t\ge 0}$ denotes an $\left(\mathcal{F}_t^{\sigma(A_\infty)}\right)$ local martingale. This last formula is exactly the decomposition formula obtained for the progressive enlargement with L.

Proof. We can first assume that M is an L^2 martingale; the general case follows by localization. Let Λ_s be an $\mathcal{F}_s$ measurable set, and take $t>s$. Then, for any bounded test function G, we have:

$$\begin{aligned}
\mathbf{E}\left(\mathbf{1}_{\Lambda_s}G(A_\infty)(M_t-M_s)\right) &= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\lambda_t(G)M_t-\lambda_s(G)M_s\right)\right)\\
&= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\langle\lambda(G),M\rangle_t-\langle\lambda(G),M\rangle_s\right)\right)\\
&= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\int_s^t \dot{\lambda}_u(G)\,d\langle M,\mu\rangle_u\right)\right)\\
&= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\int_s^t\int \lambda_u(dx)\,\rho(x,u)\,G(x)\,d\langle M,\mu\rangle_u\right)\right)\\
&= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\int_s^t d\langle M,\mu\rangle_u\rho(A_\infty,u)\right)\right).
\end{aligned}$$

But from (4.2), we have:

$$\rho(A_\infty,t) = \frac{1}{Z_t}\mathbf{1}_{\{A_\infty>A_t\}} - \frac{1}{1-Z_t}\mathbf{1}_{\{A_\infty=A_t\}}.$$

It now suffices to notice that (A_t) is constant after L and L is the first time when $A_\infty = A_t$, or in other words (for example, see [11] p. 134):

$$\mathbf{1}_{\{A_\infty>A_t\}} = \mathbf{1}_{\{L>t\}},\ \text{and}\ \mathbf{1}_{\{A_\infty=A_t\}} = \mathbf{1}_{\{L\le t\}}.$$

□

Remark 4.3. The method we have used here applies to many other situations, where the theorems of Jacod do not apply. Each time the different relationships we have just mentioned between the quantities: $\lambda_t(G), \dot{\lambda}_t(G)$, and $\lambda_t(dx), \dot{\lambda}_t(dx), \rho(x,t)$, hold, the above method and decomposition formula apply. The reader can refer to [33] for an application in the Brownian setting with A_∞ replaced with B_1, the value at 1 of the standard Brownian Motion.

We can now use the above initial enlargement of filtration to recover the formula for the progressive enlargement with L.

Corollary 4.4. *Let L be an honest time (we assume, as usual, that the assumptions **(CA)** hold). Then,*

$$\mathcal{F}_t^L \subset \mathcal{F}_t^{\sigma(A_\infty)},$$

and every $(\mathcal{F}_t)$ local martingale M is an $\left(\mathcal{F}_t^L\right)$ semimartingale and decomposes as:

$$M_t = \widetilde{M}_t + \int_0^t \mathbf{1}_{\{L>s\}} \frac{d\langle M, \mu\rangle_s}{Z_s} - \int_0^t \mathbf{1}_{\{L\leq s\}} \frac{d\langle M, \mu\rangle_s}{1 - Z_s},$$

where $\left(\widetilde{M}_t\right)_{t\geq 0}$ denotes a $\left(\mathcal{F}_t^L\right)$ local martingale.

Proof. We saw in the course of the proof of Theorem 4.2 that:

$$\mathbf{1}_{\{A_\infty > A_t\}} = \mathbf{1}_{\{L>t\}}, \text{ and } \mathbf{1}_{\{A_\infty = A_t\}} = \mathbf{1}_{\{L\leq t\}}.$$

Thus, by definition of $\mathcal{F}_t^L$, we have:

$$\mathcal{F}_t^L \subset \mathcal{F}_t^{\sigma(A_\infty)}.$$

Now, let M be a $(\mathcal{F}_t)$ martingale which is in L^2; the general case follows by localization. From Theorem 4.2

$$M_t = \widetilde{M}_t + \int_0^t \mathbf{1}_{\{L>s\}} \frac{d\langle M, \mu\rangle_s}{Z_s} - \int_0^t \mathbf{1}_{\{L\leq s\}} \frac{d\langle M, \mu\rangle_s}{1 - Z_s},$$

where $\left(\widetilde{M}_t\right)_{t\geq 0}$ denotes an $\left(\mathcal{F}_t^L\right)$ L^2 martingale. Thus, $\left(\widetilde{M}_t\right)$, which is equal to:

$$M_t - \left(\int_0^t \mathbf{1}_{\{L>s\}} \frac{d\langle M, \mu\rangle_s}{Z_s} - \int_0^t \mathbf{1}_{\{L\leq s\}} \frac{d\langle M, \mu\rangle_s}{1 - Z_s}\right),$$

is $\left(\mathcal{F}_t^L\right)$ adapted, and hence it is a $\left(\mathcal{F}_t^L\right)$ martingale which is in L^2. □

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